may not know about it. The subscription list was doubled in 1913. If it should be doubled again in 1914, the journal would become self-sustaining. **It can be done.**

THE TACTICAL PROBLEM OF STEINER.

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NOTE BY THE EDITORS.—This article illustrates the reference in the editorial of this issue concerning "papers of a somewhat more technical character in which, however, we have tried to have the technical terms explained for the benefit of the general reader." Professor Bussey has met this request most admirably.

The study of tactical configurations known as triple systems had its origin in two related problems proposed independently by T. P. Kirkman[1] and J. Steiner.[2] Kirkman's problem is to arrange fifteen school girls in parties of three for seven consecutive days' walk so that every two of the girls walk together once and only once during the seven days. There is a good account of the history of the problem with several methods of solution in Ball's *Mathematical Recreations and Essays*, 5th edition, Chapter 9.

An arrangement of a number of elements in sets of three so that every set of two is contained in one and only one set of three is called a triple system. The sets of three are called triples or triads. The problem of the fifteen school girls involves a triple system of 15 elements and 35 triads. The simplest triple system is the following well-known one of 7 elements and 7 triads. The digits 0, 1, 2, 3, 4, 5, 6 are the elements and the columns are the triads.

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 3 & 4 & 5 & 6 & 0 & 1 & 2 \end{matrix}$$

The seven elements of this triple system can be arranged in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad. The arrangement is as follows.

$$\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 2 & 3 & 4 & 5 & 6 & 0 & 1 \\ 5 & 6 & 0 & 1 & 2 & 3 & 4 \end{matrix}$$

It is well known that the nine points of inflexion of a plane cubic curve lie by threes on twelve straight lines. Four lines pass through each point of inflexion. Any two of the nine points thus determine a third, and the nine points form a

[1] *The Lady's and Gentleman's Diary*, 1850.

[2] *Journal für die reine und angewandte Mathematik*, Vol. 45, pp. 181–182.

triple system with twelve triads. The triple system of nine elements may be written as follows.

0	0	0	0	1	1	1	2	2	2	3	6
1	3	4	5	3	4	5	3	4	5	4	7
2	6	7	8	7	8	6	8	6	7	5	8

These triple systems of seven and nine elements are important in the theory of equations of the 7th and 9th degrees.[1]

The problem proposed by Steiner is as follows. It was suggested to him by an investigation of the configuration of the 28 double tangents of a plane quartic curve.

For what values of n is it possible to arrange n elements in sets of three, called triples or triads, so that every set of two elements is contained in one and only one triad? If n is a number for which there is such an arrangement in triads, are there other arrangements which cannot be obtained from it by a mere permutation of the elements? When such an arrangement has been made, is it possible to arrange the n elements in sets of four, called tetrads, so that no triad is contained in a tetrad and so that every set of three which is not a triad is contained in one and only one tetrad? When such an arrangement in tetrads has been made, is it possible to arrange the n elements in sets of five, called pentads, so that no triad or tetrad is contained in a pentad, and so that every set of four which is not a tetrad and does not contain a triad is contained in one and only one pentad? When these successive arrangements have been made, up to and including an arrangement in *k-ads*, is it possible to arrange the n elements in sets of $k + 1$, called *$(k + 1)$-ads*, so that no *l-ad*, $l \leq k$, is contained in a *$(k + 1)$-ad*, and so that every set of k elements which is not a *k-ad* and does not contain an *l-ad*, $l < k$, is contained in one and only one *$(k + 1)$-ad?*

The part of the problem that relates to triads has been completely solved.[2] This means that all of Steiner's questions have been answered. It does not mean that the last word on triple systems has been said or that all problems connected with triple systems have been solved.

Every pair of elements of a triple system determines a triad. The number of pairs that can be chosen from n elements is $\frac{1}{2}n(n - 1)$. But each triad is determined by any one of three pairs. Therefore, to obtain the number of triads, divide the number of pairs by three. The result is $\frac{1}{6}n(n - 1)$. The total number of elements in the triple system, counting duplicates, is three times the number of triads, namely $\frac{1}{2}n(n - 1)$. Therefore each of the n elements occurs $\frac{1}{2}(n - 1)$ times. This number must be an integer, and therefore n must be an odd number. It must therefore be of one of the forms $6m + 1, 6m + 3, 6m + 5$. But $\frac{1}{6}n(n - 1)$ must also be an integer. This can happen when n is $6m + 1$ or $6m + 3$, but not when n is $6m + 5$. Therefore a necessary condition that n elements can be ar-

[1] See Netto's *Theory of Substitutions* (English translation by F. N. Cole), pp. 229–239, and Netto's *Vorlesungen über Algebra*, Vol. 2, pp. 460–480.

[2] See *Encyclopédie des Sciences Mathématiques*, Vol. 1, p. 80.

ranged to form a triple system is that n be of the form $6m + 1$ or $6m + 3$. It has been proved that this condition is also sufficient. When $n = 7$ or 9, there is essentially only one arrangement of the n elements in triads. Any two arrangements differ only in notation. When $n = 13$, there are two and only two essentially different triple systems. When n is any number of the form $6m + 1$ or $6m + 3$ and is greater than 13, there are at least two essentially different triple systems.[1] The two triple systems of 13 elements are compared in two recent papers by F. N. Cole[2] and H. S. White.[3]

This paper has to do primarily with the part of the Steiner problem that relates to tetrads, pentads, etc. If an arrangement of n elements in triads and tetrads is possible, the number of tetrads can be counted as follows. Every tetrad is determined by a set of three which is not a triad. The total number of sets of three that can be chosen from n elements is $\frac{1}{6}n(n - 1)(n - 2)$. The number of these which are triads has been found to be $\frac{1}{6}n(n - 1)$. The difference between these two numbers, namely $\frac{1}{6}n(n - 1)(n - 3)$, is the number of sets of three which determine tetrads. But each tetrad can be determined by three elements in as many ways as there are combinations of four things three at a time, namely in four ways. Therefore, to get the number of tetrads, divide by four. The result is $\frac{1}{24}n(n - 1)(n - 3)$. When $n = 7$, this number is also equal to 7. An arrangement of seven elements in seven tetrads has already been given.

If there is an arrangement of n elements in triads, tetrads, pentads, etc., the number of *k-ads* for $k = 3, 4, 5$, etc., is given by the formula

$$N_k = \frac{1}{k!}n(n - 1)(n - 3)(n - 7) \cdots (n - [2^{k-2} - 1]),$$

which was given by Steiner without proof. It can be proved as follows. Let $a_1, a_2, a_3, \cdots, a_{k-1}$ be a set of $k - 1$ elements that is not a $(k - 1)$-*ad* and does not contain an l-*ad*, $l < k - 1$. Such a set determines a k-*ad*. The number of such sets is k times the number of k-*ads*, namely kN_k, because the same k-*ad* is determined by any one of k different sets of $k - 1$ elements each. Every $(k + 1)$-*ad* is determined by a set of k elements that is not a k-*ad* and does not contain an l-*ad*, $l < k$. Any $k - 1$ elements of such a set constitute a set $a_1, a_2, a_3, \cdots, a_{k-1}$ of the kind mentioned above. Therefore, to get a set of k elements to determine a $(k + 1)$-*ad*, adjoin to these an element a_k which is such that it will not combine with any of the others to form an l-*ad* for any $l \leqq k$. In choosing a_k, the following elements must be avoided.

1. The ${}_{k-1}C_1$ elements $a_1, a_2, a_3, \cdots, a_{k-1}$.
2. The ${}_{k-1}C_2$ elements that form triads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in pairs.

[1] For proofs of these statements see Moore, *Concerning Triple Systems*, *Mathematische Annalen*, Vol. 43, pp. 271–285, and Netto, *Vorlesungen über Algebra*, Vol. 2, p. 474.

[2] *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 1–5.

[3] *Transactions of the American Mathematical Society*, Vol. 14 (1913), pp. 6–13.

3. The ${}_{k-1}C_3$ elements that form tetrads with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken in threes.

. . . .

$k-1$. The ${}_{k-1}C_{k-1}$ elements that form k-*ads* with the elements $a_1, a_2, a_3, \cdots, a_{k-1}$ taken $k-1$ at a time.

The total number of elements to be avoided is therefore

$$ {}_{k-1}C_1 + {}_{k-1}C_2 + {}_{k-1}C_3 + \cdots + {}_{k-1}C_{k-1}, $$

which is the number of combinations of $k-1$ things taken any number at a time, and is equal to $2^{k-1}-1$.[1] The number of ways in which the a_k can be chosen is therefore $n-(2^{k-1}-1)$, and the number of ways in which k elements can be chosen to determine a $(k+1)$-*ad* is apparently $kN_k[n-(2^{k-1}-1)]$. But the set of k elements has been obtained by adjoining an element a_k to a certain sub-set $a_1, a_2, a_3, \cdots, a_{k-1}$. It can be obtained in the same way by adjoining an element to any other sub-set of $k-1$ of the same k elements. These sub-sets are ${}_kC_{k-1}=k$ in number. Therefore the count is k times as large as it ought to be; and the number of ways in which a set of k elements can be chosen to determine a $(k+1)$-*ad* is $N_k[n-(2^{k-1}-1)]$. But the same $(k+1)$-*ad* is determined by any k of its elements, that is in ${}_{k+1}C_k = k+1$ ways. Therefore the total number of $(k+1)$-*ads* is $N_k\left[\dfrac{n-(2^{k-1}-1)}{k+1}\right]$. In other words, to obtain the number of $(k+1)$-*ads*, multiply the number of k-*ads* by $\dfrac{n-(2^{k-1}-1)}{k+1}$. The number of tetrads has already been found to be $\dfrac{1}{4!}n(n-1)(n-3)$. Therefore the number of pentads is $\dfrac{1}{5!}n(n-1)(n-3)(n-7)$; and the number of k-*ads* for any value of k is that given by Steiner's formula.

In the *Bulletin of the American Mathematical Society*, Vol. 16, pp. 19–22, the author of this paper proved that when n is a number of the form 2^j-1 there exists an arrangement of n elements in triads, tetrads, pentads, etc., up to and including $(j+1)$-*ads*. That paper made use of finite projective geometries of many dimensions.[2] The present paper gives the solution in a more simple way which makes no use of geometry. It not only proves that the arrangement is possible, but also gives a method by which the arrangement can easily be written down in any particular case. The paper makes use of the theory of linear dependence in a finite field.

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[1] See Fine's *College Algebra*, § 770.

[2] See Veblen and Bussey, "Finite Projective Geometries," *Transactions of the American Mathematical Society*, Vol. 7, pp. 241–259.

Note.—M. G. Brunel, in *Procès-Verbaux, Soc. des Sciences de Bordeaux,* 1896–1897, pp. 37–41, gave an arrangement of 9 elements in triads and tetrads; one of 7 elements in triads and tetrads; and one of 15 elements in triads, tetrads, and pentads. He stated that the method which he used in the last two cases could be extended to the case of $2^j - 1$ elements. His paper gave merely the results in the three cases ($n = 7, 9, 15$) without any suggestion of method.

The Finite Algebra of Order p.

Any system of s distinct symbols or marks which can be combined by four operations that satisfy the same formal laws as the four fundamental operations of algebra, addition, subtraction, multiplication, and division, so that when the marks are so combined the result of the operations is in every case unique and belongs to the given system of marks, are said to constitute a "finite algebra" or "field" of order s.

In the theory of numbers the integers $0, 1, 2, \cdots, (p - 1)$ are said to constitute a system of "least residues," modulo p. No two of them are congruent to each other, and every other integer is congruent to some one of them, modulo p. The sum of two of them, say a and b, is congruent to some one of them, say c. Then c is said to be the "sum of a and b modulo p." The operation of finding c when a and b are given is called "addition modulo p." Similarly "subtraction modulo p" and "multiplication modulo p" are defined. It is proved in the theory of numbers that, when p is a prime, there is one and only one least residue c of p, such that $ac \equiv b$, modulo p, where b is any least residue of p and a is any one except zero. c is called the quotient of b divided by a modulo p, and the operation of finding c when b and a are given is called "division modulo p." Methods of finding c in any given case are given in books on the theory of numbers. It is also proved in the theory of numbers that these operations which are called addition, subtraction, multiplication, and division, modulo p, satisfy the same formal laws as the corresponding operations of ordinary algebra. Therefore the least residues modulo p, where p is a prime, constitute a "finite algebra" or "field" of order p.

Linear Dependence in the Field of Order p.

In the field of order p, m sets of n marks each

$$(A)\qquad \begin{matrix} x_1', & x_2', & x_3', & \cdots, & x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]}, & x_2^{[m]}, & x_3^{[m]}, & \cdots, & x_n^{[m]} \end{matrix}$$

are said to be linearly dependent when there exist in the field m marks $c', c'', c''', \cdots, c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \text{ mod. } p \quad (j = 1, 2, 3, \cdots, n).$$

If this is not the case, the sets are said to be linearly independent.

This definition is analogous to the definition of linear dependence in ordinary algebra.[1] The proofs of the following theorems are almost word for word like the proofs of the corresponding theorems in ordinary algebra, and for that reason are not given here.

THEOREM 1. Two sets of n marks each, not all zero, are linearly dependent when and only when they are proportional.

THEOREM 2. If there exist among m sets of marks a smaller number of sets which are linearly dependent, then the m sets are linearly dependent.

THEOREM 3. If any one of the m sets of marks consists wholly of zeros, the m sets are linearly dependent.

THEOREM 4. A necessary and sufficient condition that the m sets of marks (A) be linearly dependent when $m \leqq n$ is that all the m-rowed determinants of the matrix

$$\left\| \begin{matrix} x_1' & x_2' & x_3' & \cdots & x_n' \\ x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{matrix} \right\|$$

be congruent to zero.

THEOREM 5. m sets of n marks each are always linearly dependent when $m > n$.

LINEAR DEPENDENCE IN THE FIELD OF ORDER 2.

In the field of order 2, the only marks are 0 and 1, and therefore, by Theorem 1, two sets of n marks each, not all zero, are linearly dependent when and only when they are identical.

Consider m different sets of n marks each, not all zero, in the field of order 2, $m \leqq n$,

$$(B) \qquad \begin{matrix} x_1', & x_2', & x_3', & \cdots, & x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]}, & x_2^{[m]}, & x_3^{[m]}, & \cdots, & x_n^{[m]} \end{matrix}$$

such that the m sets are linearly dependent and such that no q of the sets are linearly dependent for any $q < m$. Then there are in the field m marks c', c'', c''', $\cdots$, $c^{[m]}$, not all zero, such that

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m]}x_j^{[m]} \equiv 0, \bmod 2 \quad (j = 1, 2, 3, \cdots, n).$$

[1] For an account of the theory of linear dependence in ordinary algebra, see Bôcher's "Introduction to Higher Algebra," Chapter III.

No one of the c's can be zero; for if one of them were, say $c^{[m]}$, this relation would become

$$c'x_j' + c''x_j'' + c'''x_j''' + \cdots + c^{[m-1]}x_j^{[m-1]} \equiv 0, \bmod 2 \quad (j = 1, 2, 3, \cdots, n),$$

where the c's are not all zero, which would prove that the first $m - 1$ sets are linearly dependent. But this is contrary to the hypothesis about the sets (B). Therefore every one of the c's must be 1, and the following theorem is proved.

THEOREM 6. *If m sets of n marks each, not all zero, in the field of order 2, $m \leqq n$, are linearly dependent, and if no q of them, $q < m$, are linearly dependent, then*

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m]} \equiv 0, \bmod 2 \quad (j = 1, 2, 3, \cdots, n).$$

THEOREM 7. *Any one of the marks $x_j^{[i]}$, $[i = 1, 2, \cdots, m]$, of Theorem (6), is congruent to the sum of all the others, modulo 2.*

This is because, in the algebra of integers modulo 2, the mark -1 is congruent to the mark $+1$, and consequently any term of a congruence can be transposed from one side of the congruence to the other.

Let the following be $m - 1$ linearly independent sets of marks of the field of order 2.

$$\begin{array}{llll} x_1', & x_2', & x_3', & \cdots, x_n' \\ x_1'', & x_2'', & x_3'', & \cdots, x_n'' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]}, & x_2^{[m-1]}, & x_3^{[m-1]}, & \cdots, x_n^{[m-1]} \end{array}$$

Adjoin to these a new set $x_1^{[m]}, x_2^{[m]}, \cdots, x_n^{[m]}$, where $x_j^{[m]}$, $(j = 1, 2, 3, \cdots, n)$, is the mark of the field of order 2 determined by the congruence

$$x_j^{[m]} \equiv x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]}, \bmod 2, \tag{1}$$

which can also be written in the form

$$x_j' + x_j'' + x_j''' + \cdots + x_j^{[m-1]} + x_j^{[m]} \equiv 0, \bmod 2.$$

This last relation proves that the m sets are linearly dependent. It can be proved that no $m - 1$ of the m sets are linearly dependent. The proof is as follows. The first $m - 1$ sets are linearly independent by hypothesis, and therefore, by Theorem 4, at least one of the $(m - 1)$-rowed determinants of the matrix

$$\left\| \begin{array}{lllll} x_1' & x_2' & x_3' & \cdots & x_n' \\ x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]} \end{array} \right\|$$

is not congruent to zero. It can be assumed without loss of generality that the sets have been so arranged that the determinant of the first $m-1$ columns is one not congruent to zero; that is, the determinant

$$(C)\qquad \begin{vmatrix} x_1' & x_2' & x_3' & \cdots & x_{m-1}' \\ x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \end{vmatrix}$$

is not congruent to zero. Now consider $m-1$ of the m sets including the m-th set, and suppose the one omitted to be the first set. The necessary and sufficient condition that they be linearly dependent is that all the $(m-1)$-rowed determinants of the matrix

$$\begin{Vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_n'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_n''' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & x_3^{[m-1]} & \cdots & x_n^{[m-1]} \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_n^{[m]} \end{Vmatrix}$$

be congruent to zero. The $(m-1)$-rowed determinant of the first $m-1$ columns is

$$(D)\qquad \begin{vmatrix} x_1'' & x_2'' & x_3'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & x_3''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ x_1^{[m]} & x_2^{[m]} & x_3^{[m]} & \cdots & x_{m-1}^{[m]} \end{vmatrix}$$

Replace each element of the last row by its value as given by relation (1), and the determinant becomes

$$\begin{vmatrix} x_1'' & x_2'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \\ (x_1'+x_1''+\cdots+x_1^{[m-1]}), & (x_2'+x_2''+\cdots+x_2^{[m-1]}), & \cdots & (x_{m-1}'+x_{m-1}''+\cdots+x_{m-1}^{[m-1]}) \end{vmatrix}$$

By the well-known theorem about the addition of determinants, this determinant is equal to the sum of $m - 1$ determinants, the first of which is

$$\begin{vmatrix} x_1'' & x_2'' & \cdots & x_{m-1}'' \\ x_1''' & x_2''' & \cdots & x_{m-1}''' \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x_1^{[m-1]} & x_2^{[m-1]} & \cdots & x_{m-1}^{[m-1]} \\ x_1' & x_2' & \cdots & x_{m-1}' \end{vmatrix}$$

The others are all congruent to zero, because in each one of them two rows are identical. The one determinant is not congruent to zero because it differs from the determinant (C) only in the order of its rows. Therefore the determinant (D) is not congruent to zero; and the $m - 1$ sets under consideration are not linearly dependent. It has now been proved that the $m - 1$ sets obtained by omitting the first of the m sets are linearly independent. The argument holds good when the omitted set is not the first one but any other one. This completes the proof that no $m - 1$ of the m sets are linearly dependent. By Theorem 2, it follows that no q of the m sets, $q < m$, are linearly dependent. The result may be stated as the following theorem.

THEOREM 8. Any $m - 1$ sets of n marks each, not all zero, in the field of order 2, which are linearly independent determine an m-th set so that the m sets are linearly dependent and so that no q of them, $q < m$, are linearly dependent.

APPLICATION TO STEINER'S PROBLEM.

Consider the $2^j - 1$ different elements of the form $(x_1, x_2, x_3, \cdots, x_j)$, each x being a mark of the field of order 2, that is, either 0 or 1, the element $(0, 0, 0, \cdots, 0)$ being excluded. Any two of them are linearly independent because they are different. Any three which are linearly dependent are said to constitute a triad. By Theorem 8, any two elements determine a third to form a triad. If the two elements are $(x_1, x_2, x_3, \cdots, x_j)$ and $(y_1, y_2, y_3, \cdots, y_j)$, the third is $(x_1 + y_1, x_2 + y_2, x_3 + y_3, \cdots, x_j + y_j)$, where the $+$ signs mean addition modulo 2. Also by Theorem 8, any three elements which are linearly independent, and therefore do not form a triad, determine a fourth element so that the four are linearly dependent but no three of them are linearly dependent. Such a set of four is called a tetrad. Any set of four elements which are linearly independent is not a tetrad and, by Theorem 2, does not contain a triad. Such a set of four determines a fifth element, by Theorem 8, such that the five are linearly dependent but also such that no four or three are linearly dependent. Such a set of five is called a pentad. Any set of k elements which are linearly independent is not a k-*ad* and, by Theorem 2, does not contain an l-*ad*, $l < k$. The k elements determine a $(k + 1)$-st element, by Theorem 8, such that the $k + 1$ elements are linearly dependent but also such that no l of them, $l < k + 1$, are linearly de-

pendent. Such a set of $k + 1$ elements is called a $(k + 1)$-*ad*. This arrangement in triads, tetrads, etc., is just the arrangement called for in the Steiner problem. There is no arrangement of $2^j + 1$ elements in *k-ads* for $k > j + 1$ because, by Theorem 5, there is no set of $j + 1$ linearly independent elements. Furthermore, the Steiner formula gives $N_k = 0$ when $k > j + 1$. But there is an arrangement in *k-ads* for every $k \leqq j + 1$.

As an example of the foregoing theory, the arrangement of seven elements in triads and tetrads is worked out as follows. In this case $j = 3$, and the $2^j - 1 = 7$ elements are

$$(001) \quad (010) \quad (101) \quad (011) \quad (111) \quad (110) \quad (100).$$

For convenience let these be denoted by the letters a, b, c, d, e, f, g, respectively. Any two elements (x_1, x_2, x_3) and (y_1, y_2, y_3) determine the third element $(x_1 + y_1, x_2 + y_2, x_3 + y_3)$, where the $+$ signs mean addition modulo 2. The three form a triad. Thus the elements a and b determine the element (001), which is the element d. Therefore one triad of the system is *abd*. Similarly all the other triads can be worked out. The complete arrangement is as follows, the columns being triads.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \\ d & e & f & g & a & b & c \end{array}$$

Let (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) denote any three elements that do not form a triad. They determine as the fourth element to form a tetrad the element $(x_1 + y_1 + z_1, x_2 + y_2 + z_2, x_3 + y_3 + z_3)$, where the $+$ signs mean addition modulo 2. Thus the three elements a (001); b (010); and c (101) determine the element (110), which is the element f. Therefore one tetrad of the system is *abcf*. The others can be found in the same way. The complete arrangement is as follows, the columns being the tetrads.

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \\ b & c & d & e & f & g & a \\ c & d & e & f & g & a & b \\ f & g & a & b & c & d & e \end{array}$$

This arrangement in triads and tetrads is, apart from notation, the same as the one given earlier in the paper.

The following table gives the number of triads, tetrads, etc., when $n = 7, 15, 31, 63$.

n.	N_3.	N_4.	N_5.	N_6.	N_7.	N_8.
7	7	7	0	0	0	0
15	35	105	168	0	0	0
31	155	1,085	5,208	13,888	0	0
63	651	9,765	109,368	974,944	3,999,744	0